

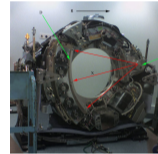
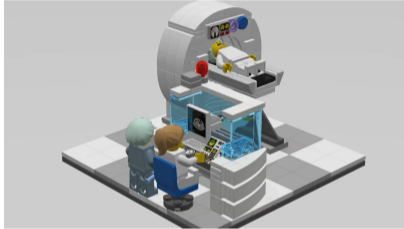
Kim Knudsen - Danmarks Tekniske Universitet

Tomografi

Hvordan vi med matematik kan se det skjulte

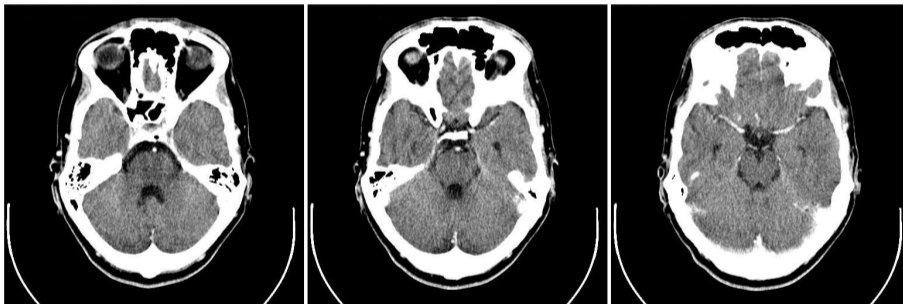
Intermat, 28. april, 2023

CT er Computer-tomografi



Græsk: Tomos = tværsnit, graph = billede

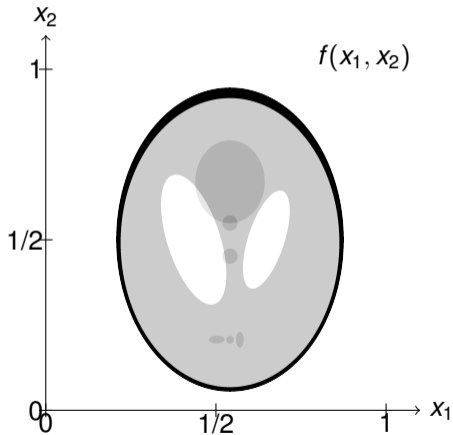
- Pioneret af Allan Cormack and Godfrey Hounsfield - Nobelpris 1979
- Matematikken henføres til Johann Radon (1917)





Billeder og matematik

Sort	10
Mørk grå	6
Lys grå	2
Hvid	0

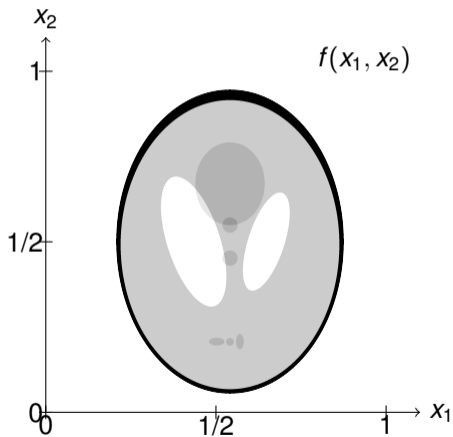


Billeder og matematik

Sort	10
Mørk grå	6
Lys grå	2
Hvid	0

$f(0.75, 0.6)$?

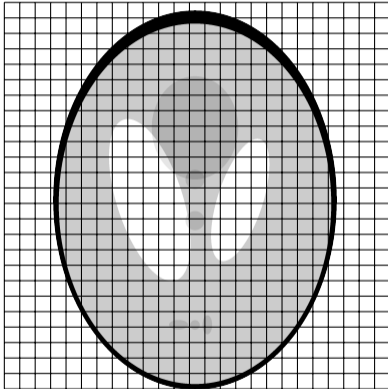
$f(0.3, 0.3)$?



Digitalt billede - zoom



Digitalt billede - matrix

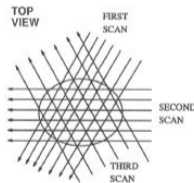
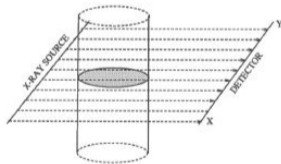


x_1	x_{N+1}	x_{2N+1}	\rightarrow	$x_{(N-1)N+1}$
x_2	x_{N+2}	x_{2N+2}	\rightarrow	$x_{(N-1)N+2}$
x_3	x_{N+3}	x_{2N+3}	\rightarrow	$x_{(N-1)N+3}$
\downarrow	\downarrow	\downarrow	\searrow	\downarrow
x_N	x_{2N}	x_{3N}	\rightarrow	x_{N^2}

Computer-tomografi

Idé:

- 1 Røngten-stråler sendes igennem objekt langs alle linier
- 2 På modsatte side måles af objektet måles strålingens intensitet
- 3 Baseret på målinger rekonstrueres et digitalt billede af de indre strukturer



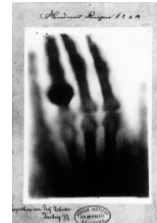
Røntgen-fysik

Røntgen stråler opdaget af Wilhelm Röntgen i 1885 (Nobelpris i 1901)

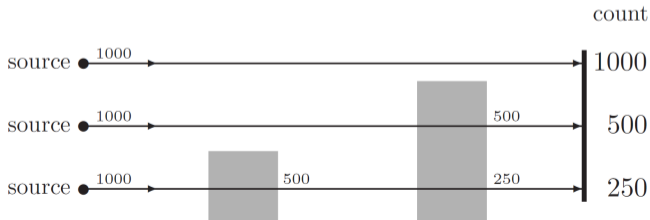


Røntgen-fysik

Røntgen stråler opdaget af Wilhelm Röntgen i 1885 (Nobelpris i 1901)



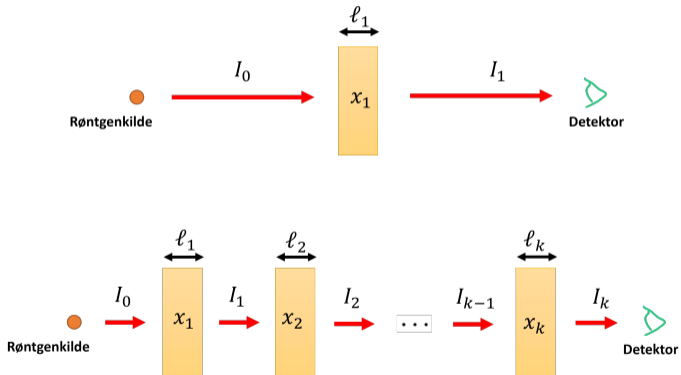
Røntgenstråling udbredes via rette linier. Absorption fører til tab af intensitet.



$$I_n = I_0 2^{-n}$$

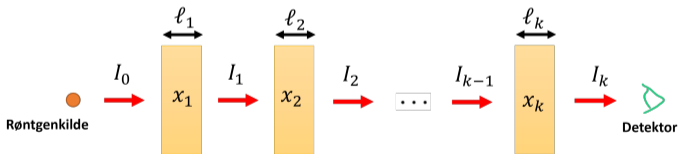
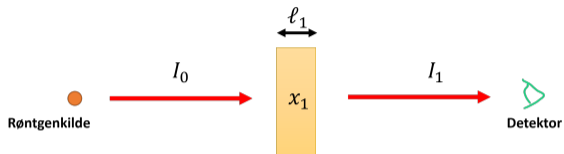
Forskellige materialer og tykkelser

Tykkelse l , absorption x



Forskellige materialer og tykkelser

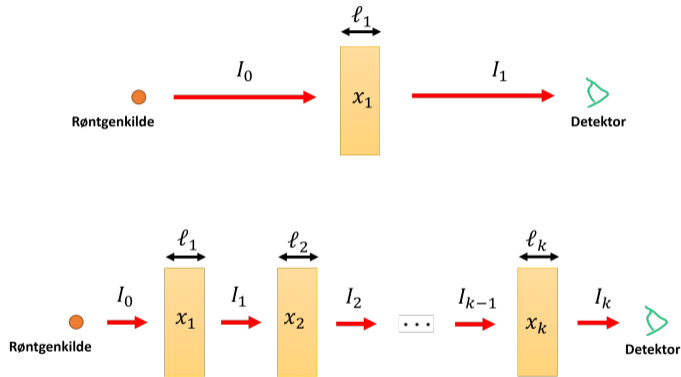
Tykkelse l , absorption x



$$I_1 = I_0 \exp(-x_1 l_1) \quad I_2 = I_1 \exp(-x_2 l_2) = I_0 \exp(-x_1 l_1) \exp(-x_2 l_2) = I_0 \exp(-x_1 l_1 - x_2 l_2)$$

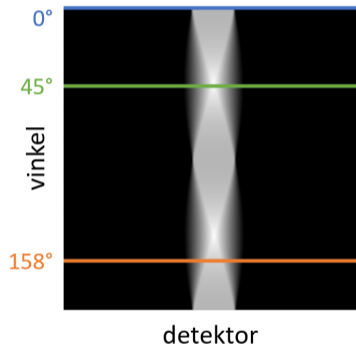
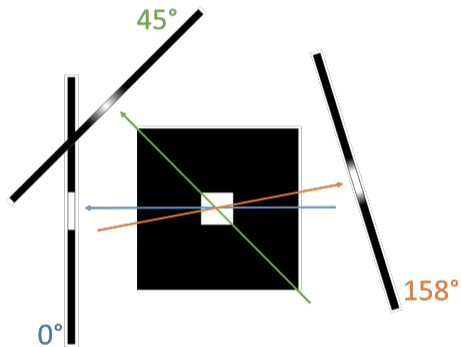
Forskellige materialer og tykkelser

Tykkelse l , absorption x



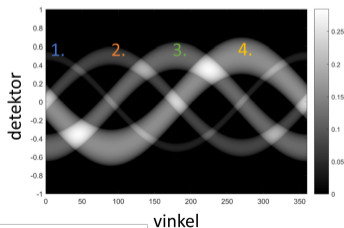
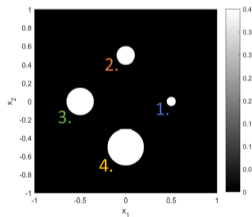
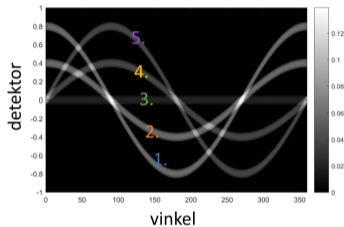
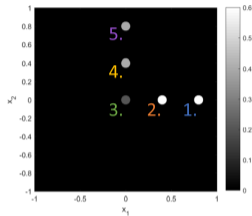
$$I_k = I_0 \exp(-x_1 l_1 - x_2 l_2 - \dots - x_k l_k) = I_0 \exp\left(-\sum_{j=1}^k x_j l_j\right) \Leftrightarrow \ln\left(\frac{I_0}{I_k}\right) = \sum_{j=1}^k x_j l_j$$

Mange målinger - Sinogram

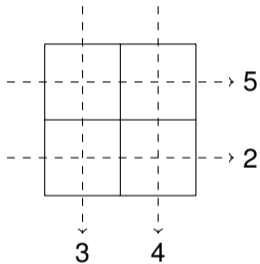


Mål: Rekonstruer billede ud fra sinogram

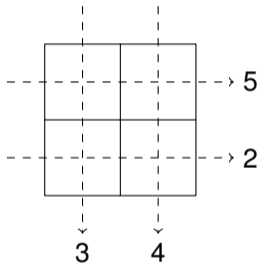
Mange målinger - Sinogram



Tal-pusleri I



Tal-pusleri I



$$x_1 + x_2 = 3 \quad x_3 + x_4 = 4$$

$$x_1 + x_3 = 5 \quad x_2 + x_4 = 2$$

Matrix-ligning

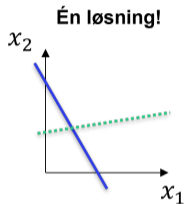
$$x_1 + x_2 = 3 \quad x_3 + x_4 = 4$$

$$x_1 + x_3 = 5 \quad x_2 + x_4 = 2$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 2 \end{pmatrix}$$

A **x** = **b**

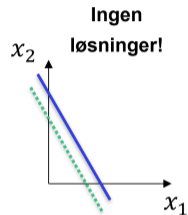
Lineære ligninger i to variable



$$\begin{aligned}2x_1 + x_2 &= 4 \\ -x_1 + 3x_2 &= 5\end{aligned}$$

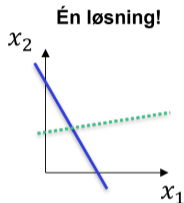


$$\begin{aligned}2x_1 + x_2 &= 4 \\ 4x_1 + 2x_2 &= 8\end{aligned}$$

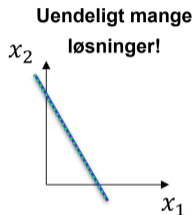


$$\begin{aligned}2x_1 + x_2 &= 4 \\ 4x_1 + 2x_2 &= 5\end{aligned}$$

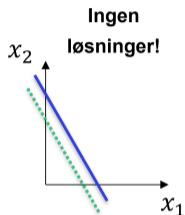
Lineære ligninger i to variable



$$\begin{aligned}2x_1 + x_2 &= 4 \\ -x_1 + 3x_2 &= 5\end{aligned}$$



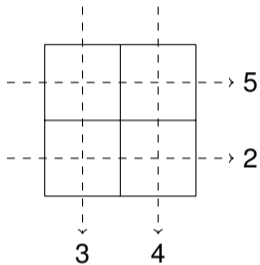
$$\begin{aligned}2x_1 + x_2 &= 4 \\ 4x_1 + 2x_2 &= 8\end{aligned}$$



$$\begin{aligned}2x_1 + x_2 &= 4 \\ 4x_1 + 2x_2 &= 5\end{aligned}$$

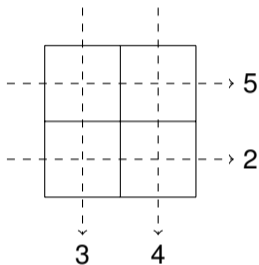
- Lineære ligninger i tre variable: 2D planer i 3D rum
- Lineære ligninger i fire variable: 3D hyperplaner i 4D rum

Fuldstændig løsning



$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

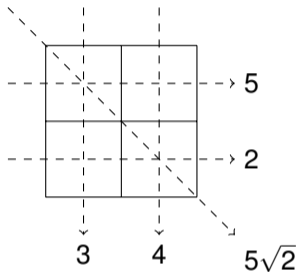
Fuldstændig løsning



$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Uendeligt mange løsninger - underbestemt system

Flere målinger og ligninger

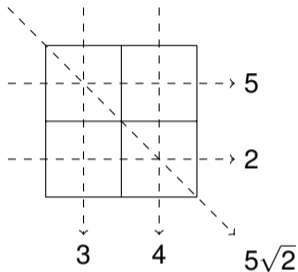


$$x_1 + x_2 = 3 \quad x_3 + x_4 = 4$$

$$x_1 + x_3 = 5 \quad x_2 + x_4 = 2$$

$$\sqrt{2}x_1 + \sqrt{2}x_4 = 5\sqrt{2}$$

Flere målinger og ligninger



$$x_1 + x_2 = 3 \quad x_3 + x_4 = 4$$

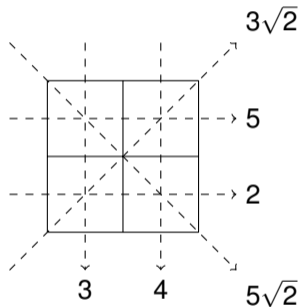
$$x_1 + x_3 = 5 \quad x_2 + x_4 = 2$$

$$\sqrt{2}x_1 + \sqrt{2}x_4 = 5\sqrt{2}$$

Entydig løsning - bestemt problem

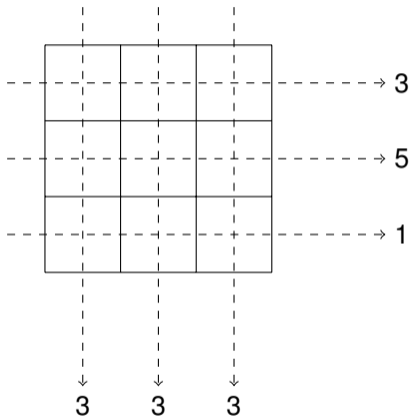
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 2 \end{pmatrix}$$

Endnu flere målinger og ligninger

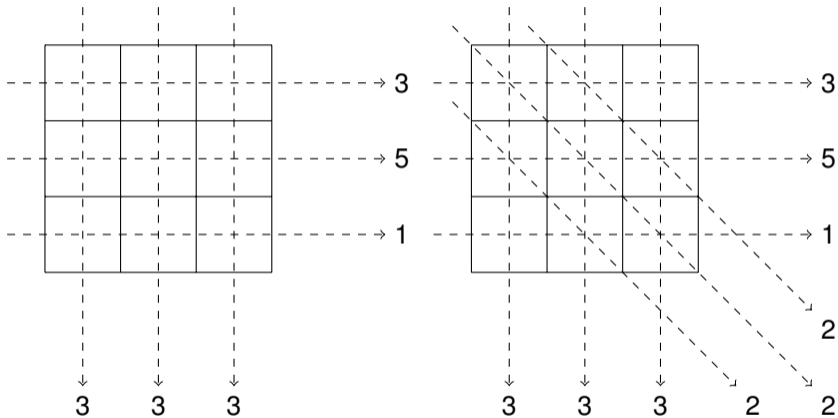


Ingen løsninger - overbestemt problem

Tal-pusleri 2

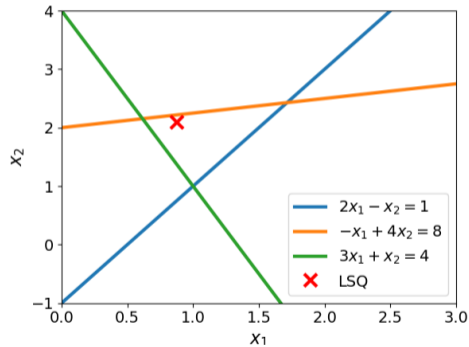


Tal-pusleri 2



6 ligninger med 9 ubekendte – 9 ligninger med 9 ubekendte.

Ingen løsninger - mindste kvadraters metode



Mindste kvadraters metode

Lineær ligning opstilles i matrix

$$\mathbf{Ax} = \mathbf{b}$$

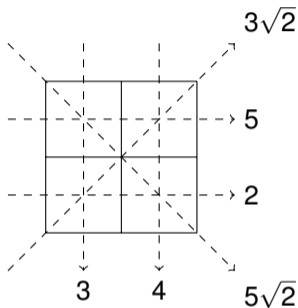
Mindste kvadraters metode:

$$\mathbf{x} = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) \quad J(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

Stationære punkter

$$\nabla J(\mathbf{x}) = 0 \Leftrightarrow (\mathbf{A}^T \mathbf{A})\mathbf{x} = \mathbf{A}^T \mathbf{b}.$$

Mindste kvadraters løsning

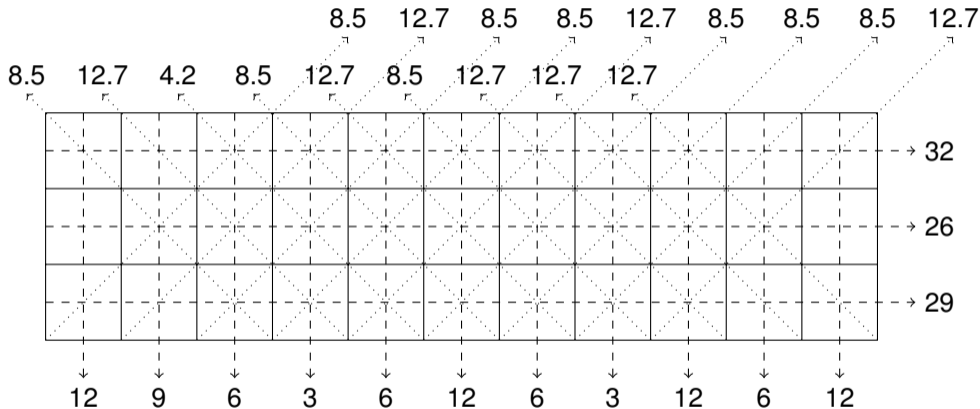


$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

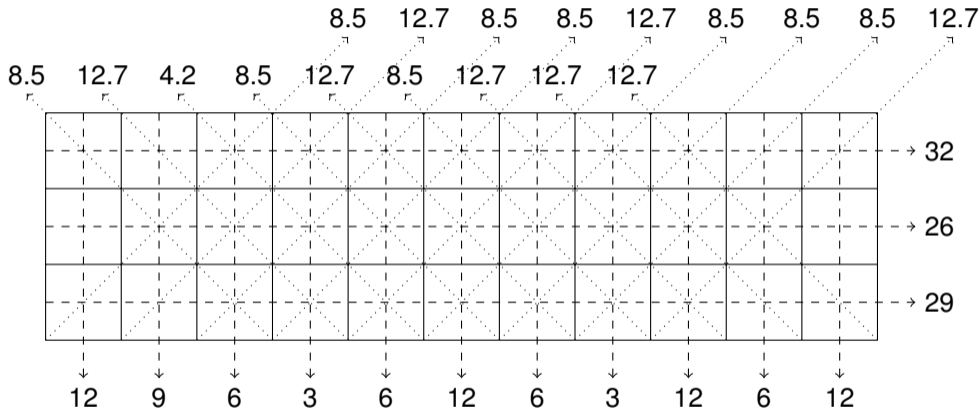
$$b = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 2 \\ 5\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

Mindste kvadraters løsning: $\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b} \Leftrightarrow \mathbf{x} = \begin{pmatrix} 2.83 \\ 0.33 \\ 2.33 \\ 1.83 \end{pmatrix}$

Grand challenge



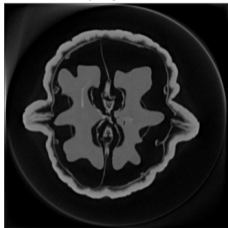
Grand challenge



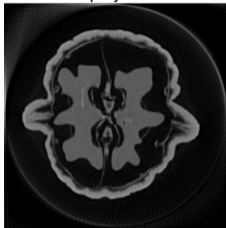
Maple!

Nye udfordringer: Få projektioner

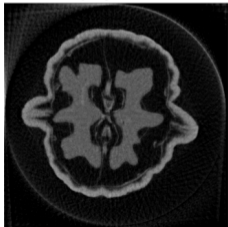
400 projektioner



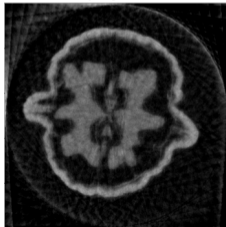
200 projektioner



100 projektioner



50 projektioner



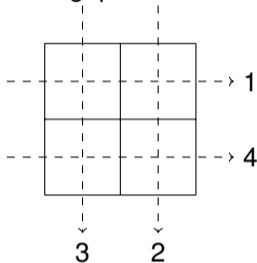
Tomografi er også

- Ultralyds-tomografi
- Elektrisk Impedans-tomografi
- Positrons-absorptions-tomografi
- Seismisk tomografi

SRP om tomografi

Opgave

Betragt pusleriet



- 1 Skriv de tilhørende fire ligninger med fire ubekendte og find en løsning.
- 2 Find den fuldstændige løsning
- 3 Tilføj ligningen $\sqrt{2}x_2 + \sqrt{2}x_3 = \sqrt{2} \cdot 2$. Hvilken gennemgående linje svarer det til? Find nu den entydige løsning.
- 4 Tilføj ligningen $x_1 + x_4 = 2$. Hvilken linje er der tale om? Find ved brug af mindste kvadraters metode en løsning (brug gerne CAS).